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A contribution to queens graphs: A substitution method[☆]

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Abstract

A graph G is a queens graph if the vertices of G can be mapped to queens on the chessboard such that two vertices are adjacent if and only if the corresponding queens attack each other, i.e. they are in horizontal, vertical or diagonal position.

We prove a conjecture of Beineke, Broere and Henning that the Cartesian product of an odd cycle and a path is a queens graph. We show that the same does not hold for two odd cycles. The representation of the Cartesian product of an odd cycle and an even cycle remains an open problem.

We also prove constructively that any finite subgraph of the rectangular grid or the hexagonal grid is a queens graph.

Using a small computer search we solve another conjecture of the authors mentioned above, saying that $K_{3,4}$ minus an edge is a minimal non-queens graph.

Keywords: Queens graph; Geometric representation; Product graph; Odd cycle

1. Introduction

Different representation problems arise naturally in (geometric) graph theory. We consider a topic which could be treated as a purely geometric problem, but for its correspondence to a chess figure, we use this latter, picturesque language.

We consider only simple and finite graphs, and use standard notations of [2]. For any edge-transitive graph G, we denote by G^- the graph arising from G by deleting one edge. We let P_k denote the simple path on k vertices.

A graph G is a *queens graph* if the vertices of G can be mapped to queens on the infinite chessboard such that two vertices are adjacent if and only if the corresponding queens attack each other ignoring the other queens, i.e. they are in horizontal, vertical or diagonal position. Such a proper placement of queens is called a *representation* or *realization* of G. In this case G is called *representable* or *realizable*.

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An important property of queens graphs is that their class is closed under taking spanned subgraphs. Thus if G is a queens graph, then so is any spanned subgraph H of G. Also if H is not representable, neither is any graph G containing H as a spanned subgraph.

The basic question is to decide whether a given graph is a queens graph or not. The difficulty of the problem is demonstrated by the following facts. If we answer the question for a given G, we cannot deduce the same answer for subgraphs of G. On the other hand, if we glue together two representable graphs with an edge, we cannot answer the question for this new graph. In view of this, Lemmas 2.9 and 2.10 are surprising, since those state that all subgraphs of a family of queens graphs are also queens graphs.

With respect to spanned subgraphs minimal non-representable graphs will be called *obstructions*. If we would like to characterize some (the) class of queens graphs, then we should list the obstructions. Determining a complete list is usually too difficult, so even an infinite list of obstructions is an achievement, see our Lemma 3.4.

We have studied queens graphs and rediscovered many of the results published in [1]. We recall some of those facts in Lemma 2.1. Using completely new ideas we extend them in several directions and prove a conjecture of Beineke, Broere and Henning that the Cartesian product of an odd cycle and a path is representable by queens on a chessboard.

At the end, we list the small obstructions for being a queens graph. These obstructions have at most seven vertices. The list shows the validity of another conjecture formulated in [1].

We use the following language to shorten the explanations. A *line* through a square S is the set of squares of the chessboard in any of the four chess directions. The set of squares to the up-right (also down-left) from a fixed square is called the *right diagonal*, and the perpendicular direction is defined as the *left diagonal*.

Let a representation R of G be given. We use the following two operations:

(i) Let σ_k be the central dilatation about the middle of a square with coefficient k. Then $\sigma_k(R)$ is another representation of G called the *k-enlargement* of R.
(ii) The *rotation* of R through $k\pi/4$ is another representation of G given by applying the coordinate transformation $(i, j) \to (i - j, i + j)$ k times.

Note that this definition is different from the usual definition of a rotation, e.g. in our case the rotation through 2π equals to σ_{16}.

We use the standard notion of Cartesian product of graphs.

Definition 1.1. Let G and H be arbitrary graphs. The Cartesian product $G \times H$ of G and H is the following. The vertices are: $(u, v) \in V(G \times H)$ if $u \in V(G)$ and $v \in V(H)$. The edges: $((u, v), (u', v')) \in E(G \times H)$ if and only if $u = u'$ and $(v, v') \in E(H)$ or $(u, u') \in E(G)$ and $v = v'$.

Finally we remark that some queens graphs has been studied extensively. The domination number and the independence number of those queens graphs that we obtain when we put a queen on each square of an $n \times m$ chessboard were in focus. In this paper we do not touch those aspects.

2. Cartesian products

There is an obvious obstruction for being a queens graph. The graph $K_{1,5}$ is not a queens graph, since the neighbors of a vertex may occupy four directions, so among five independent neighbors there must be adjacent ones. Hence a queens graph cannot have a $K_{1,5}$ as a spanned subgraph.

We recall here some relevant results of [1]. The claim in (i) states that the obvious obstruction mentioned above is the only obstruction for the class of trees.

Lemma 2.1. (i) *A tree is a queens graph if and only if it does not contain $K_{1,5}$ as an induced subgraph.*
(ii) *The following graphs are queens graphs*:

(a) $P_m \times P_k$
(b) $C_{2k} \times P_m$
(c) $C_{2k+1} \times P_2$

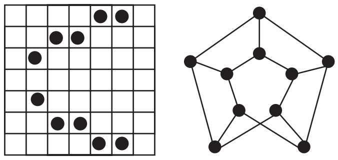

Fig. 1. Representation of the Möbius-ladder on 10 vertices.

Let the *Möbius-ladder* L_{2n} be defined as follows. Consider an even cycle C_{2n} and add to it all edges between diagonally opposite vertices (Fig. 1).

Lemma 2.2. *For any positive integer n the Möbius-ladder L_{2n} is a queens graph.*

Proof. We consider the following representation of P_n. Put the first queen arbitrarily. The relative position of the ith queen compared to the $(i-1)$th one depends on the parity of i. If i is even then make one step to the right, otherwise make one step diagonally up-right to put the next queen there. Then we repeat these two kinds of placements until we get P_n. Consider a reflection of the derived configuration respect to a horizontal line such that the image of the last point is on the left diagonal of the first point. □

We place the results of Lemma 2.1 in a more general frame. For this we need one more natural definition. A graph G is a *rooks graph* if the vertices can be mapped to rooks on the chessboard such that two vertices of G are adjacent if and only if the corresponding rooks attack each other, i.e. they are in horizontal or vertical position.

Note that any rooks graph G is also a queens graph, since the possible diagonal positions of rooks can be avoided by some shifts. This way we obtain a particular representation of G by queens with only horizontal and vertical attacks. Using a rotation through $\pi/4$ we obtain another representation where all the attacks are diagonal. We use these observations to deduce the following

Theorem 2.3. *For any rooks graphs G and H their Cartesian product $G \times H$ is a queens graph.*

Proof. Let us choose a large n and a representation of G by queens attacking only in vertical and horizontal directions on the $n \times n$ chessboard. Consider a representation of H using only diagonal attacks, and refine the chessboard by dividing each square into n^2 equally small squares. (This operation is the same as σ_n in some sense.) We call the squares of the original chessboard *big squares*; their size on the new board is $n \times n$. We replace the big square of every vertex of H with the chosen representation of G. Each queen placed has an ancestor in the chosen realization of G and another ancestor in H.

The queens placed onto the new chessboard in this way realize a graph J. We observe that two vertices of J are adjacent exactly in the following cases:

(i) They are in the same big square, and their ancestors in G are adjacent.
(ii) They are in different big squares, their ancestors in G are the same, and their ancestors in H are adjacent.

This implies that $J = G \times H$, so we obtained a representation of $G \times H$. □

Theorem 2.3 implies several results of [1], e.g. parts (a) and (b) of Lemma 2.1 (ii). We call the idea of the above proof a *substitution method*. This will lead us to settle a conjecture formulated in [1]. Before that we need a technical lemma.

In the realization of queens graphs the queens can only be placed on the squares of the chessboard. This is equivalent to placement of queens on points of the coordinate plane with integer coefficients. The next lemma shows that the integrality condition is not restrictive; we get the same class of graphs if we may place the queens anywhere on the plane.

Definition 2.4. Consider the following representation of a graph G: let the vertices be points in the plane, and two vertices are adjacent if and only if the slope of the line determined by their corresponding points is $-\infty$, -1, 0, or 1. We call this representation a slope representation of G.

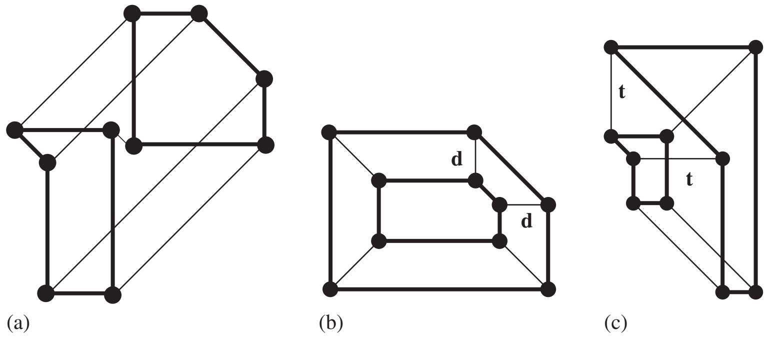

Fig. 2. Three types of links between two C_5's.

Lemma 2.5. *The graph G is a queens graph if and only if it has a slope representation.*

Proof. It is enough to prove that if a graph G has a slope representation, it can also be represented by points with only rational coordinates. Let n be the number of vertices of G. We consider the slope representation and define a set of linear equations and non-equations: if two points are adjacent, then the slope of their common line is determined, and if they are non-adjacent, then this slope may not be equal to the four given directions. All of the coefficients of those equations are 1, 0 or −1.

If a set of n points satisfies those equations and non-equations, then it is a representation of G. Since we can enlarge or translate the given representation, there are infinitely many solutions. Hence we have some free variables, and if we choose all of them to be rational numbers, we obtain a representation by points with rational coordinates. □

Theorem 2.6. *For any positive integers m and k the graphs $C_{2k+1} \times P_m$ are queens graphs.*

Proof. The case $k=1$ or $m=1$ is easy. First we construct a representation of $C_5 \times P_m$, then we extend this representation inductively. We start with $C_5 \times P_2$, Fig. 2 shows several representations of it.

We realize $C_5 \times P_m$ by constructing a chain of the two type of representations of C_5 used in Fig. 2.

The key observation is that representations (b) and (c) are possible to scale as it is pictured in Fig. 2. In the chain of C_5's let the consecutive ones to be linked using the following type pattern: (a)(b)(a)(c)(a)(b)... In the case of type (b) and (c) links the scaling parameters d and t are chosen so large that the unwanted attacks are avoided. The method can be seen in Fig. 3, where we drew the representation of $C_5 \times P_4$.

Now we describe the iteration step. Our representation of $C_{2k+1} \times P_m$ extends the representation of $C_{2k-1} \times P_m$. We distinguish one vertex of C_{2k-1} such that we obtain m distinguished vertices of $C_{2k-1} \times P_m$ spanning a P_m. In the case $k=2$ the distinguished vertices are marked with a ring in Fig. 3. Note that the distinguished vertices are always in the same position of the representation, either the concave vertex of representation (c) or the right vertex of the north side of the type (b) pentagon.

Let us divide each square of the chessboard into 25 equally small squares. If there was a queen on a given square in the representation of $C_{2k-1} \times P_m$, do the following. If the queen stands for a non-distinguished vertex, we put the same queen in the middle one of the 25 new small pieces. In the case of the distinguished queens we place three queens instead on the corresponding 5×5 subchessboards according to Fig. 4. Observe that we gave the solution for all three types of links on the same picture.

Making one step with the proper substitutions, we obtain a representation of $C_{2k+1} \times P_m$ from $C_{2k-1} \times P_m$.

Note that in the later steps we can choose any vertex of the three new ones to be a distinguished vertex, so we can continuously iterate this process. □

Unfortunately the above method does not work for $C_{2k+1} \times C_{2m}$. It would be enough to find a realization of $C_5 \times C_{2m}$ so that one C_{2m} is represented by queens attacking alternately in horizontal, right diagonal, vertical, right diagonal directions. If we had such a representation, then with the above extension method we could find a representation for $C_{2k+1} \times C_{2m}$ for any $k \geqslant 2$. However, we did not find a proper representation even of $C_5 \times C_4$.

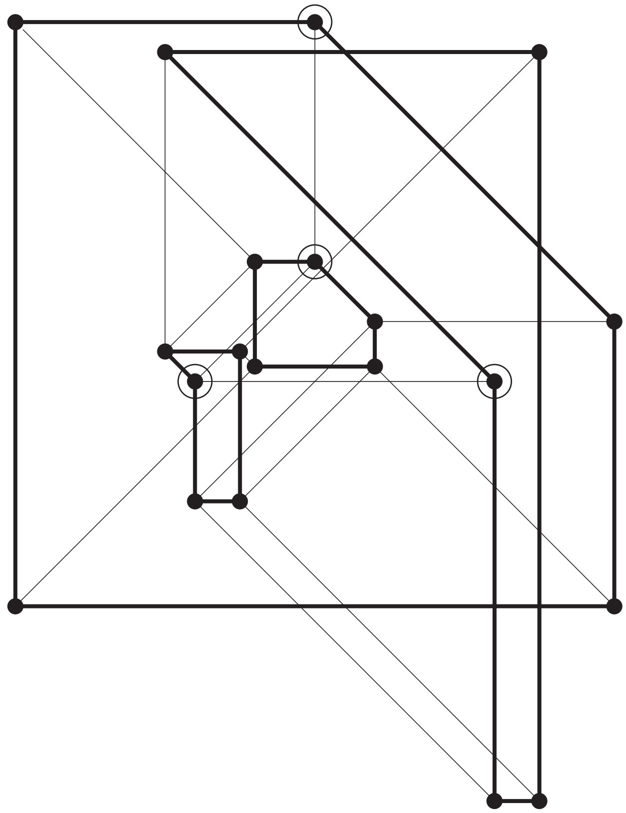

Fig. 3. The representation of $C_5 \times P_4$.

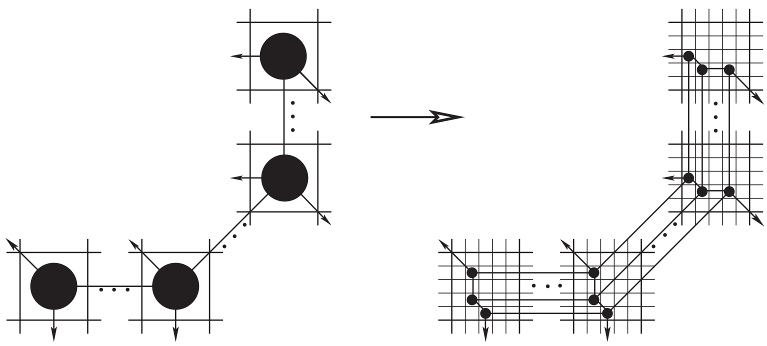

Fig. 4. The extension procedure.

The following results show that the Cartesian product of two odd cycles is not realizable.

Lemma 2.7. *If G has no triangles and its edge-chromatic number is greater than* 4, *then G is not a queens graph.*

Proof. Since G has no triangle, the neighbours of any vertex v are in different directions. We may think of the direction classes as colors. In this way a queen representation of G would correspond to a 4-edge-coloring, which is not possible by assumption. □

Corollary 2.8. *For any integers $k \geqslant m \geqslant 2$ the graph $C_{2k+1} \times C_{2m+1}$ is not a queens graph.*

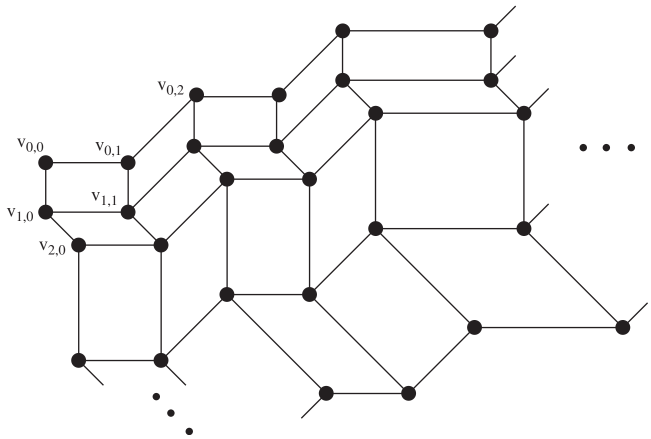

Fig. 5. The square grid.

Proof. We use the previous lemma. Since $C_{2k+1} \times C_{2m+1}$ is triangle-free, we only have to show that it is not 4-edge-colorable. Suppose to the contrary that $C_{2k+1} \times C_{2m+1}$ has a valid 4-edge-coloring. Then any color class must constitute a perfect matching. This is impossible, since we have an odd number of vertices. □

Note that the converse is not true; a 4-edge-coloring of a triangle-free graph G does not yield the queen representation of G, e.g. $K^{-}_{3,4}$ is such an example (see Fig. 9).

We remark that $C_{2k+1} \times C_{2m}$ has a 4-edge-coloring. Namely taking every second edge in all copies of C_{2m} yields a perfect matching, which can be one color class. By deleting these edges, we get disjoint copies of $C_{2k+1} \times P_2$. In them we color the edges of C_{2k+1} greedily, using the third color only once. This partial coloring uniquely extends to $C_{2k+1} \times P_2$.

Part (a) of Lemma 2.1 (ii) shows that the square grid of size $m \times n$ is representable. It is a notable queens graph in the sense that any subgraph of it is again a queens graph.

Lemma 2.9. *For any positive integers m and n, the square grid $P_m \times P_n$ and all of its subgraphs are queens graphs.*

Proof. Consider a realization of the grid shown in Fig. 5.

In this representation all squares of the grid are represented by parallelograms. The axis-parallel paths are represented by attacks alternating between two directions. Unwanted attacks can be avoided by choosing the right scaling. Let us index the vertices by the row and column number in the grid, and let $v_{0,0}$ be the vertex in the upper-left corner. Let G be an arbitrary subgraph of the grid, and suppose that G is representable. It is enough to give a method for deleting an edge e from G. We may assume that e is represented by a horizontal attack, and $e = v_{a,b-1}v_{a,b}$, where a is an even number.

Fix the origin at any vertex of the chessboard. Let $v_{j,k}$ be a vertex of G, and $q_{j,k}$ the queen representing it. Let $x_{j,k}$ and $y_{j,k}$ denote the coordinates of this queen. We define a mapping ϕ on the representation of G as follows:

$$\phi(q_{j,k}) = \begin{cases} (3x_{j,k}, 3y_{j,k}) & \text{if } j < a \text{ or } k < b, \\ (3x_{j,k} - 1, 3y_{j,k} + 1) & \text{if } j = a \text{ and } k \geqslant b, \\ (3x_{j,k} - 1, 3y_{j,k}) & \text{if } j > a \text{ and } k \geqslant b. \end{cases}$$

For $i = 1, 2, 3$ let V_i denote the subset of $V(G)$ defined by the relations in the ith row of the above definition of ϕ. Every $V_1 - V_2$ edge except e is represented by a left diagonal attack, every $V_1 - V_3$ edge is represented by a horizontal attack, and every $V_2 - V_3$ edge is represented by a vertical attack. Using this we get that the image representation is a realization of $G - e$. □

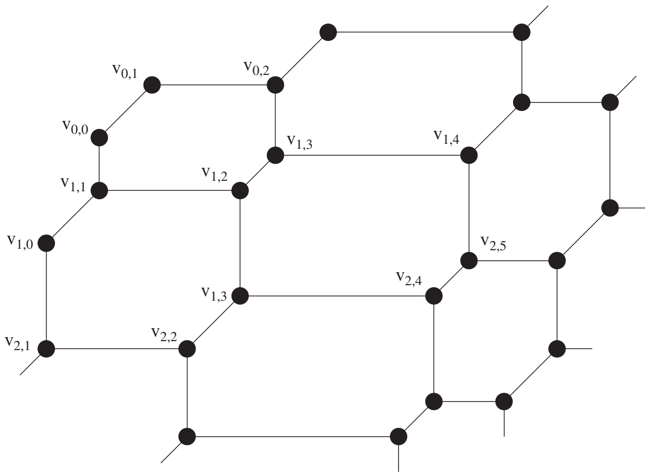

Fig. 6. The hexagonal grid.

The queens graphs considered above are planar and have maximum degree four. In principle the four directions could be enough, and any graph of maximum degree at most four could have a representation. This is not the case, see Fig. 9. However there is no such planar obstruction known. At the end of the article we conjecture a weaker claim, namely that all graphs with maximum degree three are queens graphs. As an evidence for this we give a counterpart of the previous lemma for the hexagonal grid.

Lemma 2.10. *For any positive integers m, n, the $m \times n$ hexagonal grid and all of its subgraphs are queens graphs.*

Proof. Consider the representation of the hexagonal grid $G_{m\times n}$ in Fig. 6.

We use the notations used in the proof of Lemma 2.9. Number the vertices in the manner shown in the figure, i.e. the first index denotes the row, and the second one denotes the position in the row. We give a method for deleting the edge $e = v_{a-1,b-1}v_{a,b}$ from the represented subgraph G. Define a mapping ψ on the representation of G:

$$\psi(q_{j,k}) = \begin{cases} (3x_{j,k}, 3y_{j,k}) & \text{if } j \neq a \text{ or } k < b, \\ (3x_{j,k}+1, 3y_{j,k}+1) & \text{if } j = a \text{ and } k = b, \\ (3x_{j,k}, 3y_{j,k}+1) & \text{if } j = a \text{ and } k > b. \end{cases}$$

As in the proof of Lemma 2.9, it is straightforward that the image representation is the realization of $G - e$. □

3. Concrete examples and obstructions

We prove two facts, which are useful for inductive arguments.

Lemma 3.1. *Assume G is a connected but not 2-edge-connected graph with maximum degree 4. Let e be a cut-edge and $G - e = G_1 \cup G_2$. If G_1 and G_2 are queens graphs, then G is also a queens graph.*

Proof. Let R_1 and R_2 be representations of G_1 and G_2. Assume $e = (x, y)$, $x \in G_1$ and x has no horizontal neighbor in G_1. We know that R_2 can be rotated to R_2' such that y has no horizontal neighbor in G_2. By enlarging R_2' and moving it far enough so that the representing queen of y is placed on the horizontal line of the representing queen of x, there will be no conflicts in the two representations, so we get that G is a queens graph as well. □

Lemma 3.2. *Assume G is a graph of maximum degree 3 and there is a triangle T in G. If $G - T$ is a queens graph, then so is G.*

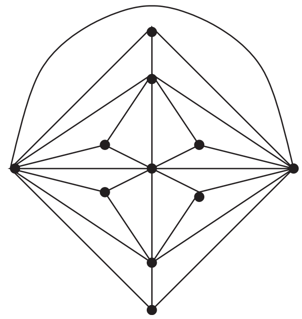

Fig. 7. The Goldner–Harary graph.

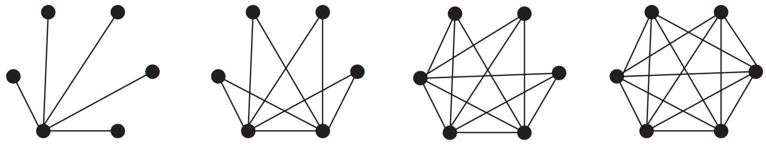

Fig. 8. The list of obstructions with six vertices.

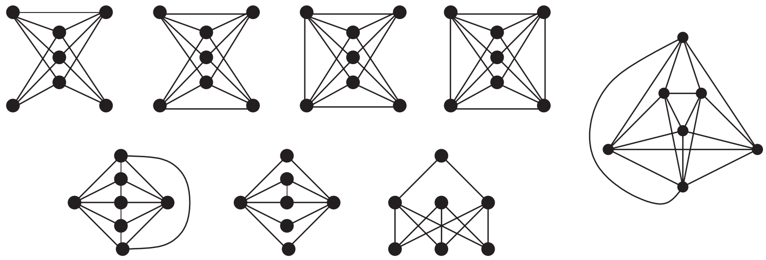

Fig. 9. The list of obstructions with seven vertices.

Proof. Since the maximum degree is 3, each vertex of T is incident with at most one edge of $E(G)\setminus E(T)$. Consider a representation of $G - T$. Look at the vertices which are adjacent to some vertex of T. There is a free direction at each of these. Intersecting these at most three lines with a line in the fourth direction, we get a triangle. If we choose the intersecting line enough far from the representation of $G - T$, we get a representation of G, so G is a queens graph as well. $\square$

We remark that the condition on the maximum degree cannot be dropped, e.g. the Goldner–Harary graph (see [3]) is triangulated, but it is not a queens graph (Fig. 7).

A systematic computer search gives the obstructions with at most seven vertices. Figs. 8 and 9 contain the output of our search. The result shows that all graphs with at most five vertices are queens graphs.

We remark here that for $n \leqslant 7$ any queens graph with n vertices has a representation on the $n \times n$ chessboard, except the empty graph on three vertices.

We prove another conjecture stated in [1].

Lemma 3.3. *The graph $K_{3,4}^-$ is a minimal non-queens graph with respect to proper subgraphs.*

Proof. It is enough to show that there is no subgraph of $K_{3,4}^-$ which is an obstruction. The obstructions with seven vertices have at least 11 edges, as many as $K_{3,4}^-$ has. On the other hand, all of the obstructions with six vertices have a vertex with degree 5, and the maximal degree of $K_{3,4}^-$ is 4. $\square$

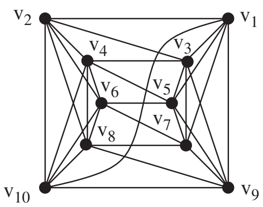

Fig. 10. The graph $C_{5,5}$.

Any odd cycle of length at least five is an obstruction for the rooks graphs. One might suspect that there are infinitely many obstructions for queens graphs as well. Let the graph $C_{n,n}$ consist of a cycle of length $2n$ with vertices $v_1, v_2, \ldots, v_{2n-1}, v_{2n}$ plus the edges between any two vertices with indices of the same parity (Fig 10).

Theorem 3.4. *The graph $C_{n,n}$ is an obstruction if $n \geqslant 5$.*

Proof. We show first that $C_{n,n}$ is not a queens graph. Suppose indirectly that there is a realization of it. Let G_1 denote the graph spanned by the vertices with odd indices, and G_2 the graph spanned by the vertices with even indices. Then $G_1 \cong G_2 \cong K_n$, and the edges between G_1 and G_2 are exactly the edges of a cycle C of length $2n$. The condition $n \geqslant 5$ implies that the only possible representation of G_1 and G_2 is n queens on a line. Observe that those two lines cannot be perpendicular.

If the two lines are parallel, we may assume that they are vertical, and v_1 is represented by a queen in the highest position. The two vertices adjacent to v_1 on C are v_2 and v_{2n}. We may suppose that v_2 is in the higher position of those two. Then the edge v_1v_2 is represented by a horizontal attack, while v_1v_{2n} by a right diagonal say, and v_2v_3 by a left diagonal. It follows that v_3 and v_{2n} are represented by queens at the same height, i.e. they are attacking, a contradiction.

The remaining possibility of the representing lines of G_1 and G_2 is that one of them is diagonal, and the other one is vertical or horizontal. In that case the edges of C are represented with attacks in the two other directions. Note that if we moved along this cycle, the coordinates of the representing queens of the vertices must change monotonously, which is not possible.

We show that every spanned subgraph of $C_{n,n}$ is a queens graph. We get those graphs by deleting some vertices from G_1 and G_2. The vertices of the subgraph are the vertices of some G_1' and G_2'. The edges are on one hand all of the edges in G_1' and G_2', and on the other hand the edges of some disjoint paths alternating between G_1' and G_2'. This graph can be represented as follows: we choose two different vertical lines for G_1' and G_2', and we choose the vertices on the suitable line so that the edges of the paths are realized by alternately horizontal and right-diagonal attacks. It is easy to check that we can avoid unwanted attacks. □

Finally we list here some of the problems we found challenging, which can serve as a base for future work.

Problem 1. Is $C_4 \times C_5$ a queens graph?

Problem 2. Is the icosahedron a queens graph? We conjecture, it is not.

Problem 3. Is it true that any (planar) graph of maximum degree 3 is a queens graph?

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